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Research on Some Fractional Integrals Based on A New Multiplication of Fractional Analytic Functions

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we find the closed forms of two types of fractional integrals by using some methods. In addition, our results are generalizations of ordinary calculus results.**

Keyword: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, closed forms, fractional integrals.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, the application of fractional derivatives and integrals has been scarce until recently. In the last decade, fractional calculus are widely used in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-12].

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [13-17]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we use some methods to obtain the closed forms of the following two types of fractional integrals:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[-r sin_{\alpha}(x^{\alpha})\otimes_{\alpha}[1+2r cos_{\alpha}(x^{\alpha})+r^{2}]^{\otimes_{\alpha}(-1)}\right],$$

and

$$\left({}_{0}I_{x}^{\alpha}\right)\left[[r cos_{\alpha}(x^{\alpha})+r^{2}]\otimes_{\alpha}[1+2r cos_{\alpha}(x^{\alpha})+r^{2}]^{\otimes_{\alpha}(-1)}\right],$$

where $0<\alpha\leq 1$, and r is a real number. Moreover, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([19])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([20])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([21])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha), g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha -1}$.

Definition 2.6 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha), g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!} (f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{12}$$

Definition 2.7 ([24])**:** If $0 < \alpha \leq 1$, and x is a real number. Then the α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes n}. \tag{13}$$

And the α-fractional cosine and α-fractional sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

Theorem 2.8 (fractional Euler's formula) ([25])**:** *If* $0 < \alpha \leq 1$,*and* $i = \sqrt{-1}$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{16}$$

III. MAIN RESULTS

In this section, we find the closed forms of two types of fractional integrals by using some techniques.

Theorem 3.1: *If* $0 < \alpha \leq 1$, *and* r *is a real number, then the* α*-fractional integrals*

$$\left({}_0I_x^\alpha \right) \left[-rsin_\alpha(x^\alpha) \otimes_\alpha [1 + 2rcos_\alpha(x^\alpha) + r^2]^{\otimes_\alpha (-1)} \right] = \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2). \tag{17}$$

And

$$\left({}_0I_x^\alpha \right) \left[[rcos_\alpha(x^\alpha) + r^2] \otimes_\alpha [1 + 2rcos_\alpha(x^\alpha) + r^2]^{\otimes_\alpha (-1)} \right] = arctan_\alpha \left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \right). \tag{18}$$

Proof Since $\left({}_0I_x^\alpha \right) \left[[1 + rE_\alpha(ix^\alpha)]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_0D_x^\alpha \right) [rE_\alpha(ix^\alpha)] \right] = Ln_\alpha(1 + rE_\alpha(ix^\alpha)).$ (19)

It follows that

$$\left({}_0I_x^\alpha \right) \left[[1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \otimes_\alpha irE_\alpha(ix^\alpha) \right] = Ln_\alpha(1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)). \tag{20}$$

And hence,

$$\left({}_0I_x^\alpha \right) \left[[1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \otimes_\alpha [-rsin_\alpha(x^\alpha) + ircos_\alpha(x^\alpha)] \right]$$
$$= \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2) + i \cdot arctan_\alpha \left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \right). \tag{21}$$

Therefore,

$$\left({}_0I_x^\alpha \right) \left[[1 + 2rcos_\alpha(x^\alpha) + r^2]^{\otimes_\alpha (-1)} \otimes_\alpha [-rsin_\alpha(x^\alpha) + ircos_\alpha(x^\alpha)] \otimes_\alpha [1 + rcos_\alpha(x^\alpha) - irsin_\alpha(x^\alpha)] \right]$$
$$= \frac{1}{2} Ln_\alpha(1 + 2rcos_\alpha(x^\alpha) + r^2) + i \cdot arctan_\alpha \left(rsin_\alpha(x^\alpha) \otimes_\alpha [1 + rcos_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \right). \tag{22}$$

Thus,

$$\left({}_{0}I_{x}^{\alpha}\right)\left[-rsin_{\alpha}(x^{\alpha})\otimes_{\alpha}[1+2rcos_{\alpha}(x^{\alpha})+r^{2}]^{\otimes_{\alpha}(-1)}\right]=\frac{1}{2}Ln_{\alpha}(1+2rcos_{\alpha}(x^{\alpha})+r^{2}).$$

And

$$\left({}_{0}I_{x}^{\alpha}\right)\left[[rcos_{\alpha}(x^{\alpha})+r^{2}]\otimes_{\alpha}[1+2rcos_{\alpha}(x^{\alpha})+r^{2}]^{\otimes_{\alpha}(-1)}\right]=arctan_{\alpha}\left(rsin_{\alpha}(x^{\alpha})\otimes_{\alpha}[1+rcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right).$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we find the closed forms of two types of fractional integrals by using some methods. Moreover, the major results we obtained are natural generalizations of the results in classical calculus. In the future, we will continue to use our methods to study the problems in engineering mathematics and fractional differential equations.

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